RANK OF A MATRIX

The RANK of a matrix is an important concept in survey network adjustment using least squares. If, in the course of a least squares adjustment of a survey network, a system of RANK DEFICIENT normal equations arises, then there is no solution to this adjustment using the conventional methods of solution of equations (matrix inverse) and the adjustment will fail.

These notes follow closely the techniques, notation and matrix algebra rules in *Observations and Least Squares* by E.M. Mikhail (Mikhail 1976) and attempt to explain rank of a matrix, rank deficiency and the connection between rank deficiency and network datum deficiency. These connections will be explained by using the common model of least squares adjustment of observation equations (parametric least squares).

Consider a set of n observation equations in u unknowns $(n > u)$ in the matrix form

$$\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f} \tag{1}$$

$\mathbf{v}$ is the $n \times 1$ vector of residuals
$\mathbf{x}$ is the $u \times 1$ vector of unknowns (or parameters)
$\mathbf{B}$ is the $n \times u$ matrix of coefficients (design matrix)
$\mathbf{f}$ is the $n \times 1$ vector of numeric terms (constants)
n is the number of equations (or observations)
u is the number of unknowns

Enforcing the least squares condition leads to the normal equations

$$\mathbf{N}\mathbf{x} = \mathbf{t} \tag{2}$$

where

$$\mathbf{N} = \mathbf{B}^T\mathbf{W}\mathbf{B} \quad \text{and} \quad \mathbf{t} = \mathbf{B}^T\mathbf{W}\mathbf{f} \tag{3}$$

$\mathbf{N}$ is the $u \times u$ coefficient matrix of the normal equations
$\mathbf{t}$ is the $u \times 1$ vector of numeric terms
$\mathbf{W}$ is the $n \times n$ weight matrix

If **N** is non-singular, i.e., $\left|\mathbf{N}\right| \neq \mathbf{0}$ and $\mathbf{N}^{-1}$ exists: there is a solution for the unknowns **x** given by matrix inversion

$$\mathbf{x} = \mathbf{N}^{-1}\mathbf{t} \tag{4}$$

If **N** is singular, i.e., $\left|\mathbf{N}\right| = \mathbf{0}$ and $\mathbf{N}^{-1}$ does not exist: there is no solution by conventional means (i.e., by the usual matrix inverse).

In least squares adjustments of survey data, a singular set of normal equations is a rank deficient set of equations and arises because the design matrix **B** is rank deficient. This is invariably due to an incorrectly posed problem, i.e., there are datum defects which could be; no coordinate origin defined, network orientation not defined or height datum not defined. The datum defects are directly connected to the rank deficiency of the design matrix **B**.

DEFINITION OF RANK OF A MATRIX AND MATRIX ALBEBRA RELATING TO RANK OF A MATRIX

The rank of a matrix is the order of the largest non-zero determinant that can be formed from the elements of the matrix by appropriate deletion of rows or columns (or both). Thus a matrix is said to be of *rank m* if and only if it has *at least one non-singular sub-matrix of order m*, but has no non-singular sub-matrix of order more than *m*.
A non-singular matrix of order *n* has a rank *n*.
A matrix with zero rank has elements that must all be zero.

1. A rectangular matrix **B** of order $n \times u$ has a rank that is less than or equal to the smallest dimension. Therefore if $u < n$ then $rank\left(\mathbf{B}\right) \leq u$. If $rank\left(\mathbf{B}\right) < u$ then **B** is said to be *rank deficient*. If a square matrix **N** is of order $u \times u$ then **N** has a rank that is less than or equal to *u*, i.e., $rank\left(\mathbf{N}\right) \leq u$.

2. The rank of the product of a number of matrices does not exceed the *least* rank of individual matrices, or

$$rank\left(\mathbf{A}_1\mathbf{A}_2\mathbf{A}_3\cdots\mathbf{A}_k\right)=\min\left[rank\left(\mathbf{A}_1\right),rank\left(\mathbf{A}_2\right),\ldots,rank\left(\mathbf{A}_k\right)\right]$$

3. Multiplication of a matrix by a non-singular matrix does not change its rank. Note here that a non-singular matrix is defined to be a square matrix whose inverse exists and a singular matrix is a square matrix whose inverse does not exits. If **B** is of order $n\times u$ $\left(n>u\right)$ and **W** is of order $n\times n$ (and $\mathbf{W}^{-1}$ exists) then the product $\mathbf{B}^T\mathbf{W}$ has order $u\times n$ and rank $\leq u$.

4. If **C** and **D** are $m\times k$ and $k\times n$ respectively, and each is of rank *k*, then **CD** is of rank *k*.

5. An important fact is that $\mathbf{BB}^T$ and $\mathbf{B}^T\mathbf{B}$ have the same rank as **B**. It follows that if **B** is of order $n\times u$ $\left(n>u\right)$ and is of (full) rank *u*, then $\mathbf{B}^T\mathbf{B}$ of order $u\times u$ is non-singular, whereas $\mathbf{BB}^T$ of order $n\times n$ is singular with rank *u*. Furthermore, $\mathbf{B}^T\mathbf{WB}$ is non-singular if **W** is also non-singular.

6. The rank of the sum of two matrices cannot exceed the sum of their ranks.

The determination of the *rank* of a matrix by direct application of the definition (which involves determinants) is not practical, particularly when dealing with anything other than simple small matrices. Instead, we may determine the rank of a matrix by (i) Singular Value Decomposition (SVD) or (ii) by *elementary row or column operations*, which do not change the order or the rank of a matrix. SVD is beyond the scope of these notes, but sufficient to say that it is a powerful set of techniques based on linear algebra that are capable of determining solutions to sets of equations that are either singular or else numerically very close to singular. For example, MATLAB has SVD routines that are used to determine matrix rank. Instead we will show how elementary row transformations can be used to determine the rank of a matrix. These operations are identical to those that could be used to perform matrix inversion or solve a system of equations by elimination.

RANK OF A MATRIX BY ROW TRANSFORMATIONS

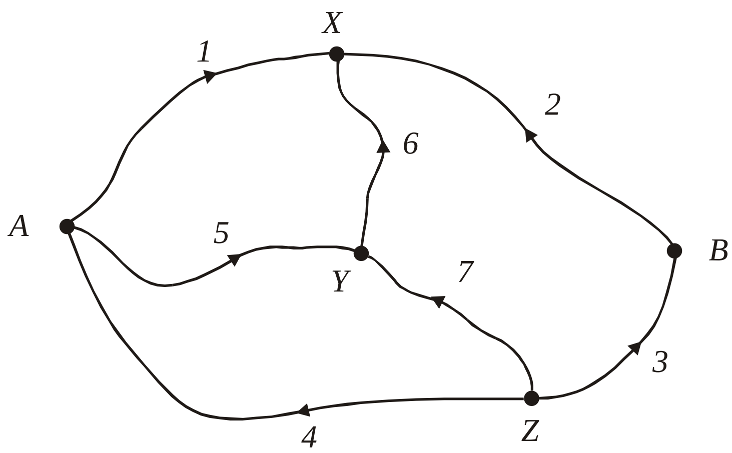
Elementary row (or column) operations do not change either the order or rank of a matrix. These operations are as follows:

1. The interchange of any two rows (or columns).
2. The multiplication of all the elements of any row (or column) by the same non-zero constant.
3. The addition (or subtraction) to any row (or column) of an arbitrary multiple of any other row (or column).

Example: The diagram shows a levelling network.

The diagram below shows a level network of height differences observed between points *A, B, X, Y* and *Z*. The arrows on the diagram indicate the direction of rise. The Table of Height differences shows the height difference for each line of the network and the distance (in kilometers) of each level run.

Line	Height Diff	Dist (km)
1	6.345	1.6
2	4.235	2.5
3	3.060	1.0
4	0.920	4.0
5	3.895	1.6
6	2.410	1.25
7	4.820	2.0



The weight of each observed height difference is inversely proportional to the distance in km's and the adjustment model is: $P + \Delta H_{PQ} + v_{PQ} = Q$ where P and Q are the heights of points, ΔH is the height difference and v is the measurement residual. All points are considered to be "floating"; hence there will be a datum deficiency and no solution by the usual means. This is because the design matrix **B** and the normal equation coefficient matrix $\mathbf{N} = \mathbf{B}^T\mathbf{WB}$ are rank deficient. The relevant matrices are:

Observation equations in the usual form $\mathbf{v} + \mathbf{B}\mathbf{x} = \mathbf{f}$ and weight matrix $\mathbf{W}$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -l_1 \\ -l_2 \\ -l_3 \\ -l_4 \\ -l_5 \\ -l_6 \\ -l_7 \end{bmatrix} \quad \text{and} \quad \mathbf{W} = \begin{bmatrix} w_1 & & & & \\ & w_2 & & & \\ & & w_3 & & \\ & & & \ddots & \\ & & & & w_7 \end{bmatrix}$$

where $w_k = \dfrac{1}{\text{dist}_k}$.

The coefficient matrix of the normal equations is

$$\mathbf{N} = \mathbf{B}^T \mathbf{W} \mathbf{B} = \begin{bmatrix} w_1 + w_4 + w_5 & 0 & -w_1 & -w_5 & -w_4 \\ 0 & w_2 + w_3 & -w_2 & 0 & -w_3 \\ -w_1 & -w_2 & w_1 + w_2 + w_6 & -w_6 & 0 \\ -w_5 & 0 & -w_6 & w_5 + w_6 + w_7 & -w_7 \\ -w_4 & -w_3 & 0 & -w_7 & w_3 + w_4 + w_7 \end{bmatrix}$$

We can see here that $\mathbf{N}$ is singular since the last row is actually the sum of the previous four rows multiplied by minus one, i.e., $row_5 = \left(row_1 + row_2 + row_3 + row_4\right) \times -1$. Or one row is a linear combination of the others and the matrix is rank deficient. Since $\mathbf{N}$ is rank deficient and therefore singular (the determinant $|\mathbf{N}| = 0$) the inverse $\mathbf{N}^{-1}$ does not exist and there can be no solution for the heights by normal means.

Substituting values for the weights gives

$$\mathbf{N} = \begin{bmatrix} 1.500 & 0.000 & -0.625 & -0.625 & -0.250 \\ 0.000 & 1.400 & -0.400 & 0.000 & -1.000 \\ -0.625 & -0.400 & 1.825 & -0.800 & 0.000 \\ -0.625 & 0.000 & -0.800 & 1.925 & -0.500 \\ -0.250 & -1.000 & 0.000 & -0.500 & 1.750 \end{bmatrix}$$

We will now use elementary row transformations to determine the rank of $\mathbf{N}$

The initial matrix **N** is

$$\mathbf{N} = \begin{bmatrix} 1.500 & 0.000 & -0.625 & -0.625 & -0.250 \\ 0.000 & 1.400 & -0.400 & 0.000 & -1.000 \\ -0.625 & -0.400 & 1.825 & -0.800 & 0.000 \\ -0.625 & 0.000 & -0.800 & 1.925 & -0.500 \\ -0.250 & -1.000 & 0.000 & -0.500 & 1.750 \end{bmatrix}$$

and the following five row transformations are performed on **N**

1.1 Multiply the elements of row 1 in **N** by $\dfrac{1}{\mathbf{N}_{11}}$. This produces a "new" row 1 and converts $\mathbf{N}_{11}$ to 1.

1.2 Multiply the elements of the new row 1 by $\mathbf{N}_{21}$ and subtract from the elements of row 2. This produces a "new" row 2 and converts $\mathbf{N}_{21}$ to zero. The multiplier is the element $\mathbf{N}_{21} = 0.000$

1.3 Multiply the elements of the new row 1 by $\mathbf{N}_{31}$ and subtract from the elements of row 3. This produces a "new" row 3 and converts $\mathbf{N}_{31}$ to zero. The multiplier is the element $\mathbf{N}_{31} = -0.625$

1.4 Multiply the elements of the new row 1 by $\mathbf{N}_{41}$ and subtract from the elements of row 4. This produces a "new" row 4 and converts $\mathbf{N}_{41}$ to zero. The multiplier is the element $\mathbf{N}_{41} = -0.625$

1.5 Multiply the elements of the new row 1 by $\mathbf{N}_{51}$ and subtract from the elements of row 5. This produces a "new" row 5 and converts $\mathbf{N}_{51}$ to zero. The multiplier is the element $\mathbf{N}_{51} = -0.250$

After performing these operations, the transformed matrix **N** is

1st transform:
$$\mathbf{N} = \begin{bmatrix} 1.0000 & 0.0000 & -0.4167 & -0.4167 & -0.1667 \\ 0.0000 & 1.4000 & -0.4000 & 0.0000 & -1.0000 \\ 0.0000 & -0.4000 & 1.5646 & -1.0604 & -0.1042 \\ 0.0000 & 0.0000 & -1.0604 & 1.6646 & -0.6042 \\ 0.0000 & -1.0000 & -0.1042 & -0.6042 & 1.7083 \end{bmatrix}$$

and the first column has been reduced to zeroes except for the leading diagonal element which is unity.

Repeating these elementary operations will successively reduce columns 2, 3, etc to zeroes excepting for leading diagonal elements, which become ones. These operations produce

2nd transform: $$\mathbf{N} = \begin{bmatrix} 1.0000 & 0.0000 & -0.4167 & -0.4167 & -0.1667 \\ 0.0000 & 1.0000 & -0.2857 & 0.0000 & -0.7143 \\ 0.0000 & 0.0000 & 1.4503 & -1.0604 & -0.3899 \\ 0.0000 & 0.0000 & -1.0604 & 1.6646 & -0.6042 \\ 0.0000 & 0.0000 & -0.3899 & -0.6042 & 0.9940 \end{bmatrix}$$

3rd transform: $$\mathbf{N} = \begin{bmatrix} 1.0000 & 0.0000 & 0.0000 & -0.7213 & -0.2787 \\ 0.0000 & 1.0000 & 0.0000 & -0.2089 & -0.7911 \\ 0.0000 & 0.0000 & 1.0000 & -0.7312 & -0.2688 \\ 0.0000 & 0.0000 & 0.0000 & 0.8892 & -0.8892 \\ 0.0000 & 0.0000 & 0.0000 & -0.8892 & 0.8892 \end{bmatrix}$$

4th transform: $$\mathbf{N} = \left[\begin{array}{cccc:c} 1.0000 & 0.0000 & 0.0000 & 0.0000 & -1.0000 \\ 0.0000 & 1.0000 & 0.0000 & 0.0000 & -1.0000 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 & -1.0000 \\ 0.0000 & 0.0000 & 0.0000 & 1.0000 & -1.0000 \\ \hdashline 0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \end{array}\right]$$

After the 4th transformation, the upper-left sub-matrix of **N** is an Identity matrix of order 4. The determinant of this sub-matrix (a determinant of order 4) is the product of the diagonal elements and is non-zero. The determinant of **N** (a determinant of order 5) will be zero since the 5th row consists entirely of zeroes and according to the definition of *rank;* the rank of **N** is 4, the order of the largest non-zero determinant. It should be noted here that the process of row transformations could be modified so that only the elements of columns below the leading diagonal were reduced to zeroes. The result would be the same: a row of zeroes after the 4th transform.

RANK DEFICIENCY AND DATUM DEFECT

Here we have shown (for the level network example) that the normal equation coefficient matrix **N** has $rank\left(\mathbf{N}\right)=4$. This is one less than its possible rank; $rank\left(\mathbf{N}\right)\le u$ where u = 5 in this example, hence we say that **N** has rank deficiency of 1. This rank deficiency also applies to **B** (the design matrix), since from our matrix algebra rules for rank we have $rank\left(\mathbf{N}=\mathbf{B}^T\mathbf{W}\mathbf{B}\right)=4$ which is the same as $rank\left(\mathbf{B}\right)=4$, since the ranks of **B** and $\mathbf{B}^T$ are identical and multiplication by a non-singular matrix **W** does not change the rank of the product. Hence **B** also has a rank deficiency of 1; the same as **N**. This rank deficiency is also the datum defect, i.e., we say that this least squares problem has a datum defect of 1. This is because we have failed to define the height datum or the origin of heights. If we assign a height to any one of the five points (*A*, *B*, *X*, *Y* or *Z*) then this datum defect will disappear. We will have defined the datum for heights and a solution of the relevant system of equations will be possible. In this example, fixing a single point will have the effect of reducing the design matrix **B** to order 7×4 (n = 7 equations in u = 4 unknowns) and it will have $rank\left(\mathbf{B}\right)=4$, as will the normal coefficient matrix **N**, and there will be a solution for the other four points.

Rank deficiency and datum defects are also considerations in 2-Dimensional (2D) and 3D survey networks. In a 2D network, if no points are held fixed (and there are measured distances) then there will be a datum defect of 3 (two translations and one rotation) and in a 3D network (again with measured distances) there will be 6 datum defects if no points are held fixed; three translations and three rotations.

REFERENCES

Mikhail, E.M., 1976, *Observations and Least Squares,* IEP–A Dun-Donnelley Publisher, New York.